حل تمرين رياضي مهندسي:

۱)جواب سؤالات بخش اعداد مختلط

الف) $Z_1\overline{Z_2}$

$Z_1 = 2i \qquad Z_2 = 1 - i \qquad Z_3 = 3 - 2i$

ب) $\dfrac{Z_1\bar{z}_1}{Z_3} = \dfrac{2i\times(-2i)}{3-2i} = \dfrac{4}{3-2i}\times\dfrac{3+2i}{3+2i} = \dfrac{12+8i}{13} = \dfrac{12}{13}+\dfrac{8}{13}i$

$\dfrac{Z_1+Z_3}{\overline{Z_2}}$

ج)

۲)

ب) $1+\sqrt{3}i = Z$

$$Z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

$r = \sqrt{x^2+y^2} = \sqrt{3+1} = 2$

$\theta = \tan^{-1}(\frac{y}{x}) = \tan^{-1}(\sqrt{3}) \Rightarrow \theta = \frac{\pi}{3}$

$\Rightarrow Z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

۳)

ب) $Z_1\overline{Z_2} = 2e^{i\pi}\times\sqrt{2}e^{-i\frac{\pi}{3}}$

$Z = re^{i\theta} \Rightarrow \bar{z} = re^{-i\theta}$

$\Rightarrow Z_1\cdot\overline{Z_2} = 2\sqrt{2}e^{i\frac{2\pi}{3}}$

۴)

$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$

$\forall n = 3 \longrightarrow (\cos\theta + i\sin\theta)^3 = (\cos 3\theta + i\sin 3\theta)$

$\cos^3\theta + 3\cos^2\theta i\sin\theta + 3i^2\sin^2\theta\cos\theta + i^3\sin^3\theta = \cos 3\theta + i\sin 3\theta$

$\cos^3\theta + 3\cos^2\theta\sin\theta - 3\sin^2\theta\cos\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$

$\cos^3\theta - 3\sin^2\theta\cos\theta = \cos 3\theta, \qquad 3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$

$\longrightarrow \cos^3\theta - 3(1-\cos^2\theta)\cos\theta = \cos 3\theta$

$\cos 3\theta = \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$\Longrightarrow \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$3(1-\sin^2\theta)\sin\theta - \sin^3\theta = \sin 3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

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الف) $\mathrm{Re}(iz) = -\mathrm{Im}(z)$

$\rightarrow i(x+iy) = ix - y$

ب) $\mathrm{Re}(z) = \mathrm{Im}(iz) \rightarrow \mathrm{Im}(iz) = i(x+iy) = xi - y$

ج)

$$\overline{iz} = -i\bar{z}$$
$$\overline{i(x+iy)} = -i \times (x - iy)$$
$$\overline{Z_1 Z_2} = \overline{Z_1}\,\overline{Z_2}$$

د)

$$\overline{\bar{Z} + 3i} = Z - 3i$$
$$\overline{(\overline{x+iy}) + 3i} = Z - 3i \rightarrow \overline{x - iy + 3i} = Z - 3i$$
$$\rightarrow \overline{x + i(-y+3)} = Z - 3i \rightarrow x - i(-y+3) = Z - 3i$$
$$\rightarrow x + iy - 3i = Z - 3i$$

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الف)

$$\mathrm{Re}(\bar{Z} - i) = 2$$
$$\mathrm{Re}(x - iy - i) = 2 \rightarrow \mathrm{Re}(x + i(-y-1)) = 2$$

$$\begin{cases} x = 2 \\ y \in R \end{cases} \qquad \{(x,y) \mid x = 2, y \in R\}$$

ب)

$$\begin{aligned}
&|Z-i| = |Z+i| \qquad\qquad \{(x,y)\ |x\in R, y=0\}\\
&|x+iy-i| = |x+iy+i|\\
&|x+i(y-1)| = |x+i(y+1)|\\
&\sqrt{x^2+(y-1)^2} = \sqrt{x^2+(y+1)^2}\\
&x^2+(y-1)^2 = x^2+(y+1)^2 \rightarrow y=0
\end{aligned}$$

۷)

$$Z_1Z_2 = 0 \qquad \begin{cases} Z_1 = x_1 + iy_1 \\ Z_2 = x_2 + iy_2 \end{cases}$$

$$Z_1Z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2) = 0$$

$$\begin{cases} x_1x_2 - y_1y_2 = 0 \\ x_1y_2 + y_1x_2 = 0 \end{cases} \qquad Z_1 = x_1 + iy_1$$

$$\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix} = 0 \Rightarrow x_2^2 + y_2^2 = 0 \Rightarrow x_2 = y_2 = 0$$

$$Z = x + iy$$

۸)

الف)

$$Z^2 + 4 = 0$$

$$Z^{1/n} = Z_0 \qquad\qquad \begin{aligned} Z &= r(\cos\theta + i\sin\theta) \\ Z_0 &= \rho(\cos\varphi + i\sin\varphi) \end{aligned}$$

$$Z_0^n = Z \rightarrow r(\cos\theta + i\sin\theta) = \rho^n(\cos\varphi + i\sin\varphi)^n = \rho^n(\cos n\varphi + i\sin n\varphi)$$

$$\rightarrow n\varphi = 2k\pi + \theta \rightarrow \varphi = \frac{2k\pi + \theta}{n}$$

$$Z^{1/n} = r^{1/n}\left[\cos(\frac{\theta + 2k\pi}{n}) + i\sin(\frac{\theta + 2k\pi}{n})\right]$$

$$Z^{m/n} = r^{m/n}\left[\cos m(\frac{\theta + 2k\pi}{n}) + i\sin m(\frac{\theta + 2k\pi}{n})\right]$$

$$\Rightarrow Z^2 + 4 = 0 \Rightarrow Z^2 = -4 \Rightarrow Z = (-4)^{1/2}$$

$$\Rightarrow Z = r\left[\cos\theta + i\sin\theta\right] = -4 \rightarrow \begin{cases} r\cos\theta = -4 \\ r\sin\theta = 0 \end{cases} \qquad \begin{cases} r\rangle 0 & \sin\theta = 0 \\ r \neq 0 & \theta = k\pi \end{cases}$$

$$\Rightarrow r = 4 \qquad , \theta = \pi$$

$$Z^{1/n} = (-4)^{1/2} = (4)^{1/2}\left[\cos(\frac{\pi + 2k\pi}{2}) + i\sin(\frac{\pi + 2k\pi}{2})\right]$$

$$\forall k = 0 \rightarrow Z_1 = 2[i] \rightarrow Z_1 = 2i$$

$$\forall k = 1 \rightarrow Z_2 = -2i$$

ب)

$$Z^4 - 2Z^2 + 4 = 0$$

$$Z^2 = t \rightarrow t^2 - 2t + 4 = 0$$

$$t_1, t_2 = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = \begin{cases} 1 + i\sqrt{3} = t_1 \\ 1 - i\sqrt{3} = t_2 \end{cases}$$

$$Z_1^2 = t_1 \Rightarrow (1 + \sqrt{3}i) = Z^2 \rightarrow Z = (1 + \sqrt{3}i)^{1/2}$$

$$Z^2+\overline{Z}^2=2\rightarrow(x+iy)^2+(x-iy)^2=2$$
$$x^2+2ixy-y^2+x^2-2ixy-y^2=2$$
$$x^2-y^2=1$$

(١٠

$$(1+Z)^n=1+\frac{n}{1!}Z+\frac{n(n-1)}{2!}Z^2+\ldots\frac{n(n-1)(n-2)\ldots(n-k+1)}{K!}Z^K+\ldots$$
$$\forall n=1\rightarrow(1+Z)^1=1+Z$$
$$\forall n=p\rightarrow(1+Z)^P=1+\frac{P}{1!}Z+\frac{P(P-1)}{2!}Z^2+\ldots+\frac{P(P-1)(P-2)\ldots(P-K+1)}{K!}Z^K+\ldots$$

حكم :

$$\forall n=p+1\rightarrow(1+Z)^{P+1}=1+\frac{(P+1)}{1!}Z+\frac{(P+1)(P)}{2!}Z^2+\ldots+$$
$$+\frac{(P+1)(P)(P-1)\ldots(P-K+2)}{K!}Z^K+\ldots$$

$$(1+Z)^{P+1}=(1+Z)^P(1+Z)=\left[1+\frac{P}{1!}Z+\frac{P(P-1)}{2!}Z^2+\ldots+\frac{P(P-1)(P-2)\ldots(P-K+1)}{K!}Z^K\right]$$
$$(1+Z)$$

$$\left[1+\frac{P}{1!}Z+\frac{P(P-1)}{2!}Z^2+\ldots+\frac{P(P-1)(P-2)\ldots(P-K+1)}{K!}Z^K+\ldots\right]+\left[Z+\frac{P}{1!}Z^2+\right.$$
$$\left.\frac{P(P-1)}{2!}Z^3+\ldots+\frac{P(P-1)(P-2)(P-K+1)}{K!}Z^{K+1}\quad\right]$$

$$1+Z(1+\frac{P}{1!})+Z^2(\frac{P(P-1)}{2!}+\frac{P}{1!})+\ldots\quad]$$
$$\frac{P^2-P+2P}{2!}=\frac{P^2+P}{2!}=\frac{P(P+1)}{2!}$$

(١٢

$$\left|(2\overline{Z}^{-}+5)(\sqrt{2}-i)\right|=\sqrt{3}|2Z+5|$$
$$\left|\underbrace{(2\overline{Z}+5)}_{Z_1}\right|\left|\underbrace{\sqrt{2}-i}_{Z_2}\right|$$

$$|Z_1Z_2|=|Z_1||Z_2|\rightarrow2\overline{Z}+5=2(x-iy)+5=2x+5-2ixy$$
$$|Z_1|=\sqrt{(2x+5)^2+4y^2}$$
$$|Z_2|=\sqrt{2+1}=\sqrt{3}$$
$$|Z_1Z_2|=\sqrt{3}\times\sqrt{(2x+5)^2+4y^2}$$
$$\sqrt{3}|2Z+5|$$
$$|2Z+5|=|2(x+iy)+5|=\sqrt{(2x+5)^2+y^2}$$

(١٣

الف)

$$Z=\frac{i}{-2-2i}\times\frac{-2+2i}{-2+2i}=\frac{-2i-2}{8}=\frac{-1}{4}-\frac{1}{4}i$$

$$\theta=\tan^{-1}(\frac{y}{x})=\tan^{-1}(\frac{-1/4}{1/4})\longrightarrow\theta=\frac{3\pi}{4}$$

$$\arg=2k\pi+\frac{3\pi}{4}$$

ب)

$$\frac{-2}{1+\sqrt{3}i}\times\frac{1-\sqrt{3}i}{1-\sqrt{3}i}=\frac{-2+2\sqrt{3}i}{4}=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$$

$$\theta=\tan^{-1}(\frac{\sqrt{3}/2}{-1/2})\Longrightarrow\theta=\frac{2\pi}{3}$$

$$\longrightarrow\arg(Z)=\theta+2K\pi=\frac{2\pi}{3}+2K\pi$$